

Positive Discrete Spectrum of the Evolutionary Operator of Supercritical Branching Walks with Heavy Tails

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Abstract

We consider a continuous-time symmetric supercritical branching random walk on a multidimensional lattice with a finite set of the particle generation centres, i.e. branching sources. The main object of study is the evolutionary operator for the mean number of particles both at an arbitrary point and on the entire lattice. The existence of positive eigenvalues in the spectrum of an evolutionary operator results in an exponential growth of the number of particles in branching random walks, called supercritical in the such case. For supercritical branching random walks, it is shown that the amount of positive eigenvalues of the evolutionary operator, counting their multiplicity, does not exceed the amount of branching sources on the lattice, while the maximal of these eigenvalues is always simple. We demonstrate that the appearance of multiple lower eigenvalues in the spectrum of the evolutionary operator can be caused by a kind of ‘symmetry’ in the spatial configuration of branching sources. The presented results are based on Green’s function representation of transition probabilities of an underlying random walk and cover not only the case of the finite variance of jumps but also a less studied case of infinite variance of jumps.

Key words: symmetric branching random walks; heavy tails; evolutionary operator; discrete spectrum; Green function

1 Introduction

Branching random walks are found to be useful in the investigation of a wide variety of applications in biology, the theory of homopolymers, population dynamics, see, e.g. [Kimmel and Axelrod, 2002; Cranston et al, 2009; Clauset, 2011; Bessonov et al, 2014] and the bibliography therein.

We consider a symmetric branching random walk (BRW) with continuous time on a lattice \mathbb{Z}^d , $d \geq 1$, assuming that, at the initial moment of time, there is a single particle in the system, which is located at some point x , and birth and death of particles occur at N lattice points x_1, x_2, \dots, x_N , called branching sources.

Continuous-time branching random walks (BRWs) with one particle generation centre on \mathbb{Z}^d and finite variance of jumps of an underlying random walk have been widely discussed, see, e.g. [Albeverio et al, 1998; Albeverio and Bogachev, 2000; Vatutin et al, 2003; Yarovaya, 2007, 2010, 2011] and the bibliography therein. The presence of a positive eigenvalue in the spectrum of the corresponding evolution operator ensures an exponential growth of the number of particles at each lattice point and on the entire lattice. BRWs with an exponential growth of the number of particles are called supercritical. By this reason, the authors of the mentioned above publications usually restricted themselves to determining only the highest eigenvalue. At the same time, in a number of situations, the information on whether a positive eigenvalue is unique or nonunique and, in the latter case, on the location of the other eigenvalues of the evolution operator may be important for analyzing the behavior of the corresponding BRW.

For example, the uniqueness of a positive eigenvalue substantially facilitates the study of the propagation of particle fronts [Molchanov and Yarovaya, 2012a,b]. However, in the presence of more than one sources on \mathbb{Z}^d , the behavior of solutions of differential equations for the moments of numbers of BRW particles is determined not only by the value of the leading positive eigenvalue but also by the mutual arrangement of the positive eigenvalues of

the evolution operator [Yarovaya, 2012]. In this connection, we are interested in obtaining conditions for the emergence of a simple isolated positive eigenvalue in the spectrum of the evolution operator with increasing the intensity of the branching sources. We also study the process of the appearance of the positive eigenvalues with further increasing the intensity of sources. We show that the appearance of eigenvalues and their multiplicity are determined not only by the intensities of sources but also by their spatial configuration. This study makes it possible to reveal the difference in the behavior of processes on a lattice and on continuous (see, e.g., [Cranston et al, 2009]) structures. Such a kind of results were obtained for the case of finite variance of jumps in [Antonenko and Yarovaya, 2015]. Here we present an approach allowing to investigate the case of infinite variance of jumps.

The behaviour of the mean number of particles can be described in terms of the evolutionary operator of a special type [Yarovaya, 2012], which is a perturbation of the generator \mathcal{A} of a symmetric random walk. In the case of equal intensities of sources this operator has the form

$$\mathcal{H}_\beta = \mathcal{A} + \beta \sum_{i=1}^N \delta_{x_i} \delta_{x_i}^T, \quad x_i \in \mathbb{Z}^d, \quad (1)$$

where $\mathcal{A} : l^p(\mathbb{Z}^d) \rightarrow l^p(\mathbb{Z}^d)$, $p \in [1, \infty]$, is a symmetric operator and $\delta_x = \delta_x(\cdot)$ denotes a column vector on the lattice taking the value one at the point x and zero otherwise. General analysis of this operator was first done in [Yarovaya, 2012].

In [Yarovaya, 2013b] it is shown how the operators of type (1) appear in BRW models and is demonstrated that the structure of its spectrum determines the asymptotic behaviour of the numbers of particles. For the analysis of the evolutionary equations for the mean number of particles, in [Yarovaya, 2013b] there was used the technique of differential equations in Banach spaces. In [Yarovaya, 2012] it is shown that \mathcal{H}_β is a linear bounded operator in every space $l^p(\mathbb{Z}^d)$, $p \in [1, \infty]$. All points of its spectrum outside the circle $C = \{z \in \mathbb{C} : |z - a(0)| \leq |a(0)|\}$ with $a(0) = \delta_0^T \mathcal{A} \delta_0$ may be only eigenvalues of finite multiplicity. This statement allowed to propose a general method for obtaining a finite set of equations defining conditions of the existence of isolated positive eigenvalues in the spectrum of the operator \mathcal{H}_β lying outside C .

In [Yarovaya, 2012] it is shown, that the perturbation of the form $\beta \sum_{i=1}^N \delta_{x_i} \delta_{x_i}^T$ of the operator \mathcal{A} may result in the emergence of positive eigenvalues of the operator \mathcal{H}_β and the multiplicity of each of them does not exceed N . However, the number of arising eigenvalues of \mathcal{H}_β in [Yarovaya, 2012] was not found. In [Yarovaya, 2015] it was announced, for the case of both finite and infinite variance of jumps, that the maximal eigenvalue of the operator \mathcal{H}_β is of unit multiplicity, that is simple, and the total multiplicity of all eigenvalues does not exceed N . This implies, in particular, that in fact the multiplicity of each eigenvalue of the operator \mathcal{H}_β does not exceed $N - 1$. For the case of finite variance of jumps this fact has been proved in [Antonenko and Yarovaya, 2015]. Below we present a unifying approach for such a kind of results, including the case of infinite variance of jumps.

The structure of the work is as follows. In Sect. 2, a formal description of BRW with N sources is reminded and the motivation of the work is explained. In Sect. 3, Theorem 1 on the conditions of existence of positive eigenvalues of the evolutionary operator is obtained. In Sect. 4, we establish the main results on the structure of the positive discrete spectrum of a supercritical BRW, Theorem 2, and prove that every supercritical BRW is weakly supercritical, Theorem 3. Notice, that Theorems 1–3 were announced in [Yarovaya, 2015]. Here we present full proofs for such a kind of results, accentuating the case of infinite variance of jumps. At last, in Sect. 5 we give an example, demonstrating the influence of ‘symmetry’ of the sources configuration on appearance of coinciding eigenvalues in the spectrum of the operator (1).

2 Preliminaries

The evolution of the system of particles in a BRW on \mathbb{Z}^d is defined by the number of particles $\mu_t(y)$ at moment t at each point $y \in \mathbb{Z}^d$ assuming that the system contains only one particle disposed at some point $x \in \mathbb{Z}^d$ at $t = 0$, i.e. $\mu_0(y) = \delta_x(y)$. Thus, the total number of particles on \mathbb{Z}^d satisfies the equation $\mu_t = \sum_{y \in \mathbb{Z}^d} \mu_t(y)$. The transition probability of the random walk in the BRW is denoted by $p(t, x, y)$. Let \mathbb{E}_x be the expectation of the total number of particles

on the condition that $\mu_0(\cdot) = \delta_x(\cdot)$. Then, the moments obey $m_n(t, x, y) := \mathbb{E}_x \mu_t^n(y)$ and $m_n(t, x) := \mathbb{E}_x \mu_t^n$, $n \in \mathbb{N}$.

Random walk is specified by a matrix $A = (a(x, y))_{x, y \in \mathbb{Z}^d}$ of transition intensities, where $a(x, y) = a(0, x - y) = a(x - y)$ for all x and y . Thus, the transition intensities are spatially homogeneous and the matrix A is symmetric. The law of walk is described in terms of the function $a(z)$, $z \in \mathbb{Z}^d$, where $a(0) < 0$, $a(z) \geq 0$ when $z \neq 0$ and $a(z) \equiv a(-z)$. We assume that $\sum_{z \in \mathbb{Z}^d} a(z) = 0$, and that the matrix A is irreducible, i.e. for all $z \in \mathbb{Z}^d$ there exists a set of vectors $z_1, z_2, \dots, z_k \in \mathbb{Z}^d$ such that $z = \sum_{i=1}^k z_i$ and $a(z_i) \neq 0$ for $i = 1, 2, \dots, k$.

We use the function b_n , $n \geq 0$, where $b_n \geq 0$ for $n \neq 1$, $b_1 \leq 0$, and $\sum_n b_n = 0$, to describe branching at a source. Branching occurs at a finite number of sources, x_1, \dots, x_N , and is given by the infinitesimal generating function $f(u) = \sum_{n=0}^{\infty} b_n u^n$ such that $\beta_r = f^{(r)}(1) < \infty$ for all $r \in \mathbb{N}$. The quantity $\beta_1 = f'(1)$ characterizes the intensity of a source and is denoted further by β . The sojourn time of a particle at every source is distributed exponentially with the parameter $-(a(0) + b_1)$, see [Yarovaya, 2009]. The finiteness of all moments is used in the proof of limit theorems about the behavior of the number of particles by the method of moments [Shohat and Tamarkin, 1943]. In what follows, it suffices to assume only the existence of β .

By $p(t, x, y)$ we denote the transition probability of a random walk. This function is implicitly determined by the transition intensities $a(x, y)$ (see, for example, [Gikhman and Skorokhod, 2004; Yarovaya, 2007]). Then, Green's function of the operator \mathcal{A} can be represented as a Laplace transform of the transition probability $p(t, x, y)$:

$$G_\lambda(x, y) := \int_0^\infty e^{-\lambda t} p(t, x, y) dt, \quad \lambda \geq 0.$$

The analysis of BRWs essentially depends on whether the value of $G_0 = G_0(0, 0)$ is finite or infinite. If the variance of jumps is finite, that is,

$$\sum_{z \in \mathbb{Z}^d} |z|^2 a(z) < \infty, \quad (2)$$

where $|z|$ is the Euclidian norm of the vector z , then $G_0 = \infty$ for $d = 1, 2$, and $G_0 < \infty$ for $d \geq 3$ (see, for example, [Yarovaya, 2007]). If, for all $z \in \mathbb{Z}^d$ with sufficiently large norm, the asymptotic relation

$$a(z) \sim \frac{H\left(\frac{z}{|z|}\right)}{|z|^{d+\alpha}}, \quad \alpha \in (0, 2), \quad (3)$$

holds, where $H(\cdot)$ is a continuous positive function symmetric on the sphere $\mathbb{S}^{d-1} = \{z \in \mathbb{R}^d : |z| = 1\}$, then $G_0 = \infty$ for $d = 1$ and $\alpha \in [1, 2)$ and G_0 is finite if $d = 1$ and $\alpha \in (0, 1)$ or $d \geq 2$ and $\alpha \in (0, 2)$ [Yarovaya, 2013a]. Condition (3), unlike (2), leads to the divergence of the series $a(z)$ and, thereby, to the infinity of the variance of jumps.

The analysis of the BRW model with one branching source in [Albeverio et al, 1998; Bogachev and Yarovaya, 1998; Yarovaya, 2007, 2010] showed that the asymptotic behaviour of the mean number of particles at arbitrary point as well as on the entire lattice is determined by the structure of the spectrum of the linear operator (1) when $N = 1$. In (1) the bounded self-adjoint operator \mathcal{A} in Hilbert space $l^2(\mathbb{Z}^d)$ is a generator of random walk, and $\beta \Delta_{x_1}$ specifies the mechanism of branching at the source x_1 . Let us note that the operator \mathcal{A} is generated by the matrix A of transition intensities. This model has been generalized in [Yarovaya, 2012], in particular, to the case of N sources.

The transition probability $p(t, \cdot, y)$ is treated as a function $p(t)$ in $l^2(\mathbb{Z}^d)$ depending on time t and the parameter y . Then according to [Yarovaya, 2007, 2012] we can rewrite the evolution equation as the following differential equation in space $l^2(\mathbb{Z}^d)$:

$$\frac{dp}{dt} = \mathcal{A}p, \quad p(0) = \delta_y,$$

where the operator \mathcal{A} acts as

$$(\mathcal{A}u)(z) := \sum_{z' \in \mathbb{Z}^d} a(z - z')u(z').$$

In the same way we can obtain the differential equation in space $l^2(\mathbb{Z}^d)$ for the expectation $m_1(t, \cdot, y)$ which can be considered as a function $m_1(t)$ in $l^2(\mathbb{Z}^d)$:

$$\frac{dm_1}{dt} = \mathcal{H}_\beta m_1, \quad m_1(0) = \delta_y. \quad (4)$$

Formally, this equation holds for $m_1(t) = m_1(t, \cdot)$ on the condition that $m_1(0) = 1$ in space $l^\infty(\mathbb{Z}^d)$.

It follows from the general theory of linear differential equations in Banach spaces (see, for example, [Daletski and Krein, 1970]) that the investigation of behaviour of solutions of the equation (4) can be reduced to the analysis of the spectrum of the linear operators in the right-hand sides of the corresponding equations. Spectral analysis of the operator \mathcal{H}_β of type (1) was done in [Yarovaya, 2012].

3 Positive Eigenvalues of the Evolutionary Operator

We denote by β_c the minimal value of intensity of sources such that the spectrum of the operator \mathcal{H}_β contains positive eigenvalues for $\beta > \beta_c$ sufficiently close to β_c . By the requirement of minimality the quantity β_c is defined uniquely.

Theorem 1. *Suppose that a BRW is based on a symmetric spatially homogeneous irreducible random walk and one of conditions (2) or (3) holds.*

If $G_0 = \infty$, then $\beta_c = 0$ for $N \geq 1$. If $G_0 < \infty$, then $\beta_c = G_0^{-1}$ for $N = 1$ and $0 < \beta_c < G_0^{-1}$ for $N \geq 2$.

To prove Theorem 1 we will need some auxiliary facts.

Lemma 1. *The quantity $\lambda > 0$ is an eigenvalue of the operator \mathcal{H}_β if and only if at least one of the equalities*

$$\gamma_i(\lambda)\beta = 1, \quad i = 0, \dots, N-1, \quad (5)$$

holds, where $\gamma_i(\lambda)$ are the eigenvalues of the matrix $\Gamma(\lambda)$ given by the equality

$$\Gamma(\lambda) = [\Gamma_{ij}(\lambda)], \quad (6)$$

with the elements $\Gamma_{ij}(\lambda) = G_\lambda(x_i, x_j) > 0$, where x_1, x_2, \dots, x_N are branching sources.

The idea of reducing, under appropriate conditions, the infinite-dimensional eigenvalue problem to a finite-dimensional one is not new, see, e.g. [Kato, 1957; Kuroda, 1963; Sadovnichii and Lyubishkin, 1986; Arazy and Zelenko, 1999]. Lemma 1 can be derived from a more general statement, Theorem 6 from [Yarovaya, 2012], but for the sake of completeness of exposition we prefer to give below its full proof.

Proof. The quantity $\lambda > 0$ is an eigenvalue of the operator $\mathcal{H}_\beta = \mathcal{A} + \beta \sum_{i=1}^N \delta_{x_i} \delta_{x_i}^T$ if and only if the following equation holds for some vector $h \neq 0$:

$$\mathcal{A}h + \beta \sum_{i=1}^N \delta_{x_i} \delta_{x_i}^T h = \lambda h.$$

Let $R_\lambda = (\mathcal{A} - \lambda I)^{-1}$ be the resolvent of the operator \mathcal{A} . By applying R_λ to both sides of the last equation we obtain

$$h + \beta \sum_{i=1}^N R_\lambda \delta_{x_i} \delta_{x_i}^T h = 0.$$

Since $\delta_x \delta_x^T h = \delta_x(\delta_x, h)$, then

$$h + \beta \sum_{i=1}^N (\delta_{x_i}, h) R_\lambda \delta_{x_i} = 0.$$

Let us scalar left-multiply the last equation by δ_{x_k} :

$$(\delta_{x_k}, h) + \sum_{i=1}^n \beta (\delta_{x_i}, h) (\delta_{x_k}, R_\lambda \delta_{x_i}) = 0, \quad k = 1, \dots, n.$$

By denoting $U_k = (\delta_{x_k}, h)$ we then obtain

$$U_k + \sum_{i=1}^n \beta U_i (\delta_{x_k}, R_\lambda \delta_{x_i}) = 0, \quad k = 1, \dots, n. \quad (7)$$

Thus, the initial equation has a nonzero solution h if and only if the determinant of the matrix of the obtained linear system is equal to zero. Now we notice that

$$(\delta_y, R_\lambda \delta_x) = -\frac{1}{(2\pi)^d} \int_{[-\pi, \pi]^d} \frac{e^{i(\theta, y-x)}}{\lambda - \phi(\theta)} d\theta,$$

where $\phi(\theta) = \sum_{z \in \mathbb{Z}^d} a(z) e^{i(\theta, z)}$ with $\theta \in [-\pi, \pi]^d$ is the Fourier transform of the function of transition probabilities $a(z)$. The right-hand side of the equation can be represented [Yarovaya, 2007] in terms of Green's function:

$$G_\lambda(x, y) := \int_0^\infty e^{-\lambda t} p(t, x, y) dt = \frac{1}{(2\pi)^d} \int_{[-\pi, \pi]^d} \frac{e^{i(\theta, y-x)}}{\lambda - \phi(\theta)} d\theta. \quad (8)$$

Hence $(\delta_y, R_\lambda \delta_x) = -G_\lambda(x, y)$. It implies that the condition of vanishing of the determinant of the linear system (7) can be rewritten as $\det(\beta \Gamma(\lambda) - I) = 0$, which is equivalent to equation (23) when $\beta \neq 0$.

Notice, that representation (8) implies the inequalities $\Gamma_{ij}(\lambda) = G_\lambda(x_i, x_j) > 0$, that is the matrix $\Gamma(\lambda)$ is positive.

Recalling that the eigenvalues of the matrix $\Gamma(\lambda)$ are denoted by $\gamma_i(\lambda)$, where $i = 0, \dots, N-1$, we obtain that (23) holds for some β and λ if and only if (5) is true. The lemma is proved. \square

Lemma 2. *Let $Q = (q_{ij})$ be a matrix with elements*

$$q_{ij} = \frac{1}{(2\pi)^d} \int_{[-\pi, \pi]^d} q(\theta) e^{i(\theta, x_i - x_j)} d\theta,$$

where x_1, x_2, \dots, x_N is a set of linearly independent vectors, and $q(\theta) \geq q_* > 0$ is an even function summable on $[-\pi, \pi]^d$. Then Q is a real, symmetric and positive-definite matrix satisfying $(Qz, z) \geq q_*(z, z)$.

Proof. The matrix Q is real and symmetric since the function $q(\theta)$ is even and then

$$q_{ij} = \frac{1}{(2\pi)^d} \int_{[-\pi, \pi]^d} q(\theta) \cos(\theta, x_i - x_j) d\theta.$$

Thus, we need to prove only that the matrix Q is positive-definite. By definition, $(Qz, z) = \sum_{i,j=1}^N q_{ij} z_i z_j$, where $z = (z_1, z_2, \dots, z_N)$. Then

$$\begin{aligned} (Qz, z) &= \frac{1}{(2\pi)^d} \sum_{i,j=1}^N \int_{[-\pi, \pi]^d} q(\theta) e^{i(\theta, x_i - x_j)} z_i z_j d\theta \\ &= \frac{1}{(2\pi)^d} \sum_{i=1}^N \sum_{j=1}^N \int_{[-\pi, \pi]^d} q(\theta) \left(e^{i(\theta, x_i)} z_i \right) \left(e^{-i(\theta, x_j)} z_j \right) d\theta \\ &= \frac{1}{(2\pi)^d} \int_{[-\pi, \pi]^d} q(\theta) \sum_{i=1}^N \sum_{j=1}^N \left(e^{i(\theta, x_i)} z_i \right) \left(e^{-i(\theta, x_j)} z_j \right) d\theta \\ &= \frac{1}{(2\pi)^d} \int_{[-\pi, \pi]^d} q(\theta) \left| e^{i(\theta, x_1)} z_1 + \dots + e^{i(\theta, x_N)} z_N \right|^2 d\theta \geq 0. \end{aligned}$$

Since $q(\theta) \geq q_*$, then

$$\begin{aligned} (Qz, z) &\geq \frac{q_*}{(2\pi)^d} \int_{[-\pi, \pi]^d} \left| e^{i(\theta, x_1)} z_1 + \dots + e^{i(\theta, x_N)} z_N \right|^2 d\theta \\ &= \frac{q_*}{(2\pi)^d} \int_{[-\pi, \pi]^d} \left(z_1^2 + \dots + z_N^2 + \sum_{i \neq j} \left(e^{i(\theta, x_i)} z_i \right) \left(e^{-i(\theta, x_j)} z_j \right) \right) d\theta. \end{aligned}$$

The integral of the summands $\left(e^{i(\theta, x_i)} z_i\right) \left(e^{-i(\theta, x_j)} z_j\right)$ vanishes, since $x_i \neq x_j$ for $i \neq j$. Then the integral in the right-hand side of the equation can be calculated explicitly and is equal to $q_*(z_1^2 + \dots + z_N^2)$. So, $(Qz, z) \geq q_*(z, z)$. The lemma is proved. \square

Lemma 3. *Each of the functions $\gamma_i(\lambda)$ is strictly decreasing when $\lambda \geq 0$. Moreover, the total number of solutions $\lambda_i(\beta)$ of the equations $\gamma_i(\lambda)\beta = 1$, $i = 0, \dots, N-1$, does not exceed N , that is the number of eigenvalues of the operator (1) does not exceed N .*

Proof. Let the eigenvalues $\gamma_i(\lambda)$ of the matrix (6) be arranged in decreasing order:

$$0 \leq \gamma_{N-1}(\lambda) \leq \dots \leq \gamma_1(\lambda) \leq \gamma_0(\lambda).$$

Let us consider the matrix

$$\Gamma(\lambda_1, \lambda_2) := \Gamma(\lambda_1) - \Gamma(\lambda_2).$$

Taking

$$q(\theta) = \frac{1}{\lambda - \phi(\theta)}$$

we obtain for $\lambda > 0$ that

$$q(\theta) \geq \frac{1}{\lambda + s} > 0,$$

where $s = \max_{\theta \in [-\pi, \pi]^d} \{-\phi(\theta)\} > 0$. Hence, by Lemma 2 for each $\lambda > 0$ the matrix $\Gamma(\lambda)$ defined by (6) and (8) is real, symmetric and positive-definite.

From (6) and (8) we obtain also that the elements of the matrix $\Gamma(\lambda_1, \lambda_2)$ are as follows

$$\Gamma_{ij}(\lambda_1, \lambda_2) = (\lambda_2 - \lambda_1) \frac{1}{(2\pi)^d} \int_{[-\pi, \pi]^d} \frac{e^{i(\theta, x_i - x_j)}}{(\lambda_1 - \phi(\theta))(\lambda_2 - \phi(\theta))} d\theta.$$

For the continuous function

$$q(\theta) = \frac{1}{(\lambda_1 - \phi(\theta))(\lambda_2 - \phi(\theta))}$$

we have the lower bound

$$q(\theta) \geq q_*(\lambda_1, \lambda_2) := \frac{1}{(\lambda_1 + s)(\lambda_2 + s)} > 0.$$

Hence, again by Lemma 2 the matrix $\Gamma(\lambda_1, \lambda_2)$ is self-adjoint and positive-definite when $\lambda_2 > \lambda_1$. In this case the Weyl theorem [Horn and Johnson, 1985, Thm. 4.3.1] implies, for all $i = 0, \dots, N-1$, the inequalities

$$\gamma_i(\lambda_1) - \gamma_i(\lambda_2) \geq q_*(\lambda_1, \lambda_2) > 0,$$

since the minimal eigenvalue of the matrix $\Gamma(\lambda_1, \lambda_2)$ has the lower bound equal to $q_*(\lambda_1, \lambda_2)$. So, $\gamma_i(\lambda_1) > \gamma_i(\lambda_2)$ when $\lambda_2 > \lambda_1$, that is the function $\gamma_i(\lambda)$ is strictly decreasing with respect to λ .

Since the functions $\gamma_i(\lambda)$ are strictly decreasing then each of the equations $\gamma_i(\lambda)\beta = 1$, where $i = 0, \dots, N-1$, for each β has no more than one solution (the eigenvalue of the operator \mathcal{H}_β). So, the total amount of the eigenvalues of the operator \mathcal{H}_β does not exceed N . The lemma is proved. \square

Now, we are ready to prove Theorem 1.

Proof of Theorem 1. For the case when condition (2) holds the theorem is proved in [Antonenko and Yarovaia, 2015]. So, we will consider only the case when condition (3) is fulfilled.

From the integral representation (8) of the function $\phi(\theta)$ it follows that

$$c_1 \|\theta\|^\alpha \leq |\phi(\theta)| \leq c_2 \|\theta\|^\alpha \quad (9)$$

for some nonzero real constants c_1 and c_2 [Yarovaia, 2013a, 2014; Kozyakin, 2015]. Then, for $\lambda = 0$, we have

$$G_0 = \int_0^\infty p(t) dt = \frac{1}{(2\pi)^d} \int_{[-\pi, \pi]^d} \frac{d\theta}{(-\phi(\theta))} \leq \frac{1}{(2\pi)^d} \int_{[-\pi, \pi]^d} \frac{d\theta}{c_1 |\theta|^\alpha}.$$

Here, all the integrals converge when $\alpha < d$ and diverge when $\alpha \geq d$. If $G_\lambda(x, y) \rightarrow \infty$ as $\lambda \rightarrow 0$ then $\|\Gamma(\lambda)\| \rightarrow \infty$ and the principal eigenvalue of matrix $\Gamma(\lambda)$ tends to infinity as $\lambda \rightarrow 0$, $\gamma_0(\lambda) \rightarrow \infty$. Hence in this case for all $\beta > 0$ the equation $\gamma_0(\lambda)\beta = 1$ has a solution (with respect to λ) and by definition of β_c we have that $\beta_c = 0$.

Let now $G_0 < \infty$, then $G_0(x, y) < \infty$ for all x and y . So, in this case $\|\Gamma(0)\| < \infty$ and, moreover, $\Gamma(\lambda) \rightarrow \Gamma(0)$ as $\lambda \rightarrow 0$. Then there exists $\gamma_* < \infty$ such that $\gamma_0(\lambda) \leq \gamma_* < \infty$ for all λ . In this case the equation $\gamma_0(\lambda)\beta = 1$ does not have solutions (with respect to λ) as $\beta \rightarrow 0$. By Lemma 1 in this case the operator \mathcal{H}_β does not have eigenvalues when β is small, i.e. $\beta_c > 0$.

It remains to prove the upper bound for β_c . Let β be an arbitrary value of the parameter such that the operator \mathcal{H}_β has a positive eigenvalue λ . Then by Lemma 1 we have $\gamma_0(\lambda)\beta = 1$, and hence

$$\beta = \frac{1}{\gamma_0(\lambda)}. \quad (10)$$

By Lemma 1 the matrix $\Gamma(\lambda)$ is positive. Then the Perron-Frobenius theorem [Horn and Johnson, 1985, Thm. 8.2.11] implies that the principal eigenvalue $\gamma_0(\lambda)$ of the matrix $\Gamma(\lambda)$ has a corresponding eigenvector $x(\lambda)$ with all positive coordinates.

Let us represent the matrix $\Gamma(\lambda)$ as

$$\Gamma(\lambda) = G_\lambda(0, 0)I + B(\lambda),$$

where $B(\lambda) = \Gamma(\lambda) - G_\lambda(0, 0)I$. Then, in the case $N \geq 2$, all elements of the matrix $B(\lambda)$ are non-negative while its off-diagonal elements are positive. Therefore, by definition of the eigenvector $x(\lambda)$ the following equalities hold:

$$0 = \Gamma(\lambda)x(\lambda) - \gamma_0(\lambda)x(\lambda) = (G_\lambda(0, 0) - \gamma_0(\lambda))x(\lambda) + B(\lambda)x(\lambda).$$

Hence, when $\lambda = 0$,

$$0 = \Gamma(0)x(0) - \gamma_0(0)x(0) = (G_0(0, 0) - \gamma_0(0))x(0) + B(0)x(0).$$

The vector $x(0)$ has positive coordinates and therefore the vector $B(0)x(0)$ also has positive coordinates. So, the last equation holds only if $G_0(0, 0) - \gamma_0(0) < 0$. Then by (10) we obtain

$$\beta_c = \frac{1}{\gamma_0(0)} < \frac{1}{G_0(0, 0)} = \frac{1}{G_0}.$$

For $N = 1$ the critical value β_c can be found from the equation $\beta_c G_0 = 1$ and equals $\beta_c = \frac{1}{G_0}$. \square

4 Structure of the Positive Discrete Spectrum

If there exists $\varepsilon_0 > 0$ such that the operator \mathcal{H}_β has a simple positive eigenvalue $\lambda(\beta)$ when $\beta \in (\beta_c, \beta_c + \varepsilon_0)$ and this eigenvalue satisfies a condition $\lambda(\beta) \rightarrow 0$ as $\beta \downarrow \beta_c$, then we call supercritical BRW *weakly supercritical* when β is close to β_c [Yarovaya, 2015].

In connection with this definition, the question naturally arises *whether every supercritical BRW is weakly supercritical?* The affirmative answer to this question is given in Theorem 3 below which is a straightforward corollary of the following stronger statement.

Theorem 2. *Let the transition intensities $a(z)$ satisfy (2) or (3), and let $N \geq 2$. Then the operator \mathcal{H}_β may have no more than N positive eigenvalues $\lambda_i(\beta)$ of finite multiplicity when $\beta > \beta_c$, and*

$$\lambda_0(\beta) > \lambda_1(\beta) \geq \dots \geq \lambda_{N-1}(\beta) > 0,$$

where the principal eigenvalue $\lambda_0(\beta)$ has unit multiplicity. Besides there is a value $\beta_{c_1} > \beta_c$ such that for $\beta \in (\beta_c, \beta_{c_1})$ the operator \mathcal{H}_β has a single positive eigenvalue, $\lambda_0(\beta)$.

Corollary 1. *Under the conditions of Theorem 2 the multiplicity of each of the eigenvalues $\lambda_1(\beta), \dots, \lambda_{N-1}(\beta)$ does not exceed $N - 1$.*

Theorem 3. *Every supercritical BRW is weakly supercritical as $\beta \downarrow \beta_c$.*

Before start proving Theorem 2 we establish some further properties of the matrix $\Gamma(\lambda)$, see definition (6).

Lemma 4. For all $\lambda > 0$ the elements $\Gamma_{ij}(\lambda)$ of the matrix $\Gamma(\lambda)$ are continuous, decreasing in λ and satisfy

$$0 < \Gamma_{ij}(\lambda) < \Gamma_{11}(\lambda) = \dots = \Gamma_{NN}(\lambda), \quad i \neq j. \quad (11)$$

Proof. The fact that the elements $\Gamma_{ij}(\lambda) = G_\lambda(x_i, x_j)$ are continuous, positive and decreasing in λ immediates from (8). To prove inequalities (11) let us observe that by (8) we can write

$$\Gamma_{ij}(\lambda) = G_\lambda(x_i, x_j) = \frac{1}{(2\pi)^d} \int_{[-\pi, \pi]^d} \frac{\cos(\theta, x_j - x_i)}{\lambda - \phi(\theta)} d\theta. \quad (12)$$

Here, for $i \neq j$, we have $x_j \neq x_i$ and then $\cos(\theta, x_j - x_i) < 1$ on a subset of $[-\pi, \pi]^d$ of positive measure, which implies

$$\Gamma_{ij}(\lambda) = G_\lambda(x_i, x_j) < \frac{1}{(2\pi)^d} \int_{[-\pi, \pi]^d} \frac{d\theta}{\lambda - \phi(\theta)} = \Gamma_{11}(\lambda) = \dots = \Gamma_{NN}(\lambda).$$

The lemma is proved. \square

The next lemma gives a bit more precise characterization of the localization of the eigenvalues of the matrix $\Gamma(\lambda)$. Let us observe that by Lemma 3 each of the eigenvalues $\gamma_i(\lambda)$ of the matrix $\Gamma(\lambda)$ is strictly decreasing when $\lambda \geq 0$. Hence, for each $i = 0, 1, \dots, N-1$ there exists $\lim_{\lambda \rightarrow 0} \gamma_i(\lambda)$, finite or infinite, which will be denoted as $\gamma_i(0)$:

$$\gamma_i(0) := \lim_{\lambda \rightarrow 0} \gamma_i(\lambda), \quad i = 0, 1, \dots, N-1.$$

Lemma 5. Let the transition intensities $a(z)$ satisfy (2) or (3). Then there exists $\gamma^* > 0$, $\gamma^* < \gamma_0$, such that

$$0 \leq \gamma_{N-1}(0) \leq \dots \leq \gamma_1(0) \leq \gamma^* < \infty. \quad (13)$$

Proof. By Lemma 4 the function $\Gamma_{11}(\lambda)$ ($\Gamma_{11}(\lambda) = \Gamma_{22}(\lambda) = \dots = \Gamma_{NN}(\lambda)$) has a limit $\lim_{\lambda \rightarrow 0} \Gamma_{11}(\lambda)$. Our further proof will depend on whether this limit is finite or infinite.

First, let

$$\Gamma_{11}(0) := \lim_{\lambda \rightarrow 0} \Gamma_{11}(\lambda) < \infty.$$

By Lemma 4 in this case each element $\Gamma_{ij}(\lambda)$ has a finite positive limit as $\lambda \rightarrow 0$, $\lambda > 0$. This means that there exists a positive matrix $\Gamma(0)$ such that

$$\Gamma(\lambda) \rightarrow \Gamma(0) \quad \text{as } \lambda \rightarrow 0, \lambda > 0.$$

Then by the Perron-Frobenius theorem [Horn and Johnson, 1985, Thm. 8.2.11]

$$0 \leq \gamma_{N-1}(0) \leq \dots \leq \gamma_1(0) < \gamma_0$$

and we can take $\gamma^* = \gamma_1(0)$.

Now, let us consider the case

$$\Gamma_{11}(0) := \lim_{\lambda \rightarrow 0} \Gamma_{11}(\lambda) = \infty.$$

Then also

$$\Gamma_{ii}(0) := \lim_{\lambda \rightarrow 0} \Gamma_{ii}(\lambda) = \infty, \quad i = 1, 2, \dots, N.$$

In this case we hardly can make use of Perron-Frobenius theorem because the limiting matrix $\Gamma(0)$ is ‘infinite’. So, we will behave differently.

Denote by $\mathbf{1}$ the $N \times N$ matrix, all elements of which are 1’s. Then, by denoting

$$G_\lambda := \Gamma_{11}(\lambda) = \dots = \Gamma_{NN}(\lambda)$$

we can represent the matrix $\Gamma(\lambda)$, for $\lambda > 0$, as follows

$$\Gamma(\lambda) = G_\lambda \mathbf{1} + \tilde{\Gamma}(\lambda), \quad (14)$$

where

$$\tilde{\Gamma}(\lambda) = \begin{bmatrix} 0 & \Gamma_{12}(\lambda) - G_\lambda & \dots & \Gamma_{1N}(\lambda) - G_\lambda \\ \Gamma_{21}(\lambda) - G_\lambda & 0 & \dots & \Gamma_{2N}(\lambda) - G_\lambda \\ \dots & \dots & \dots & \dots \\ \Gamma_{N1}(\lambda) - G_\lambda & \Gamma_{N2}(\lambda) - G_\lambda & \dots & 0 \end{bmatrix}.$$

Direct calculations show that the symmetric matrix $G_\lambda \mathbf{1}$ has the simple eigenvalue $(N-1)G_\lambda$ and the eigenvalue 0 of multiplicity $N-1$.

Now, consider the matrix $\tilde{\Gamma}(\lambda)$. For its elements, by (12) we have

$$\Gamma_{ij}(\lambda) - G_\lambda = \frac{1}{(2\pi)^d} \int_{[-\pi, \pi]^d} \frac{\cos(\theta, x_j - x_i)}{\lambda - \phi(\theta)} d\theta - \frac{1}{(2\pi)^d} \int_{[-\pi, \pi]^d} \frac{1}{\lambda - \phi(\theta)} d\theta.$$

Then

$$\begin{aligned} |\Gamma_{ij}(\lambda) - G_\lambda| &\leq \frac{1}{(2\pi)^d} \int_{[-\pi, \pi]^d} \frac{|\cos(\theta, x_j - x_i) - 1|}{\lambda - \phi(\theta)} d\theta \\ &\leq \frac{2}{(2\pi)^d} \int_{[-\pi, \pi]^d} \frac{\sin^2 \frac{(\theta, x_j - x_i)}{2}}{\lambda - \phi(\theta)} d\theta \leq \frac{|x_j - x_i|^2}{2(2\pi)^d} \int_{[-\pi, \pi]^d} \frac{|\theta|^2}{\lambda - \phi(\theta)} d\theta. \end{aligned}$$

Therefore,

$$\limsup_{\lambda \rightarrow 0} |\Gamma_{ij}(\lambda) - G_\lambda| \leq \frac{|x_j - x_i|^2}{2(2\pi)^d} \int_{[-\pi, \pi]^d} \frac{|\theta|^2}{|\phi(\theta)|} d\theta. \quad (15)$$

In the case (2), as was shown in [Yarovaya, 2007], the function $\phi(\theta)$ satisfies the estimation $|\phi(\theta)| \geq C|\theta|^2$ for $\theta \in [-\pi, \pi]^d$. In this case condition (15) takes the form

$$\limsup_{\lambda \rightarrow 0} |\Gamma_{ij}(\lambda) - G_\lambda| \leq \frac{|x_j - x_i|^2}{2C(2\pi)^d} \int_{[-\pi, \pi]^d} \frac{|\theta|^2}{|\theta|^2} d\theta = \frac{|x_j - x_i|^2}{2C} < \infty.$$

In the case (3) due to (9) the function $\phi(\theta)$ satisfies the estimation $|\phi(\theta)| \geq C|\theta|^\alpha$ for $\theta \in [-\pi, \pi]^d$, where $\alpha \in (0, 2)$. In this case condition (15) takes the form

$$\limsup_{\lambda \rightarrow 0} |\Gamma_{ij}(\lambda) - G_\lambda| \leq \frac{|x_j - x_i|^2}{2C(2\pi)^d} \int_{[-\pi, \pi]^d} \frac{|\theta|^2}{|\theta|^\alpha} d\theta,$$

where the integral in the right-hand part is converging for all α which satisfy $\alpha - 2 < d$, and hence for all $\alpha \in (0, 2)$.

So, under conditions (2) and (3) we have the estimate

$$\limsup_{\lambda \rightarrow 0} |\Gamma_{ij}(\lambda) - G_\lambda| \leq C^*, \quad i, j = 1, 2, \dots, N, \quad (16)$$

for all $\lambda > 0$ sufficiently close to zero, where

$$C^* = \max_{i,j} \frac{|x_j - x_i|^2}{2C(2\pi)^d} \int_{[-\pi, \pi]^d} \frac{|\theta|^2}{|\theta|^\alpha} d\theta, \quad \alpha \in (0, 2].$$

Hence, for such values of $\lambda > 0$ the norm of the matrix $\tilde{\Gamma}(\lambda)$ is uniformly bounded by some constant, and consequently its spectral radius $\rho(\tilde{\Gamma}(\lambda))$ (i.e. the maximal absolute value of its eigenvalues) is also uniformly bounded by some constant γ^* , that is

$$\rho(\tilde{\Gamma}(\lambda)) \leq \gamma^* \quad (17)$$

for all $\lambda > 0$ sufficiently close to zero.

Now, observe that all matrices in (14) are symmetric and then by the Weyl theorem [Horn and Johnson, 1985, Thm. 4.3.1] the eigenvalues of the matrix $\Gamma(\lambda)$ differ from the corresponding eigenvalues of the matrix $G_\lambda \mathbf{1}$ by no more than γ^* , that is

$$|\gamma_0(\lambda) - (N-1)G_\lambda| \leq \gamma^*, \quad (18)$$

$$0 \leq \gamma_{N-1}(\lambda) \leq \dots \leq \gamma_1(\lambda) \leq \gamma^*. \quad (19)$$

Passing to the limit in (19), we obtain the required estimate $\gamma_1(0) \leq \gamma^*$. The lemma is proved. \square

Remark 1. For the case (2) of finite variance of jumps the statement of Lemma 5 was presented in [Antonenko and Yarovaya, 2015; Yarovaya, 2015]. Here we give not only the proof for this case but also for the case (3) of infinite variance of jumps.

Remark 2. Sometimes it might be useful to know explicit estimates for the ‘spectral gap’ γ^* .

As was shown in the proof of Lemma 5, in the case $\lim_{\lambda \rightarrow 0} \Gamma_{11}(\lambda) < \infty$ the quantity γ^* can be estimate as follows: $\gamma^* := \gamma_1(0)$. Define

$$M := \max_{i,j} \Gamma_{ij}(0), \quad m := \min_{i,j} \Gamma_{ij}(0),$$

then by Lemma 4 $m < M$, and the Hopf theorem [Horn and Johnson, 1985, Sect. 8.2.12] implies the following estimate for the so-called ‘spectral gap’:

$$\gamma^* = \gamma_1(0) \leq \frac{M-m}{M+m} \gamma_0(0) < \gamma_0(0).$$

In the case $\lim_{\lambda \rightarrow 0} \Gamma_{11}(\lambda) = \infty$, under conditions (2) or (3), due to (17) the quantity γ^* can be defined as $\gamma^* := \limsup_{\lambda \rightarrow 0} \rho(\tilde{\Gamma}(\lambda))$. But since for the elements of the matrix $\tilde{\Gamma}(\lambda)$ the limiting estimate (16) holds, then by [Horn and Johnson, 1985, Corollary 6.1.5] $\limsup_{\lambda \rightarrow 0} \rho(\tilde{\Gamma}(\lambda)) \leq (N-1)C^*$ and thus $\gamma^* \leq (N-1)C^*$.

Proof of Theorem 2. By Lemma 1 the eigenvalues of the operator \mathcal{H}_β satisfy the equations (5). The quantities $\gamma_i(\lambda)$ are the eigenvalues of the positive and symmetric matrix $\Gamma(\lambda)$, and (as is shown in [Molchanov and Yarova, 2012b] and also follows from Lemma 3) this matrix is the Gramian matrix for some appropriate scalar product and hence is positive-definite. In this case by Lemma 4 all eigenvalues $\gamma_i(\lambda)$ are real and positive (and can be arranged in ascending order):

$$0 \leq \gamma_{N-1}(\lambda) \leq \dots \leq \gamma_1(\lambda) \leq \gamma_0(\lambda). \quad (20)$$

From (8) it follows that elements of the matrix $\Gamma(\lambda)$ tend to zero as $\lambda \rightarrow \infty$ and so $\gamma_i(\lambda) \rightarrow 0$ as $\lambda \rightarrow \infty$ for all $i = 0, 1, \dots, N-1$. By Rellich theorem [Kato, 1966, Ch. 2, Thm. 6.8] all functions $\gamma_i(\lambda)$ are piecewise smooth when $\lambda \geq 0$.

By Lemma 4 the elements of the matrix $\Gamma(\lambda)$ are strictly positive when $\lambda > 0$ and then by Perron-Frobenius theorem [Horn and Johnson, 1985, Thm. 8.2.11] the principal eigenvalue $\gamma_0(\lambda)$ of the matrix $\Gamma(\lambda)$ has the unit multiplicity and strictly exceeds other eigenvalues, i.e. the last of the inequalities (20) is strict:

$$0 \leq \gamma_{N-1}(\lambda) \leq \dots \leq \gamma_1(\lambda) < \gamma_0(\lambda). \quad (21)$$

Again by Lemma 4 the matrix $\Gamma(\lambda)$ is continuous for all values $\lambda > 0$. Behaviour of this matrix can differ as $\lambda \rightarrow 0$ and further proof of the theorem depends on this behaviour.

As we noted earlier, the quantity $\Gamma_{11}(\lambda) = \dots = \Gamma_{NN}(\lambda) = G_\lambda(0, 0)$ has a limit as $\lambda \rightarrow 0$, finite or infinite. If the quantity $G_\lambda(0, 0)$ tends to some finite limit as $\lambda \rightarrow 0$ then by Lemma 5 the eigenvalues $\gamma_i(\lambda)$ behave as is shown in Fig. 1, where $\gamma_1(0) = \frac{1}{\beta_{c1}} \leq \gamma^*$. In the case when the transition intensities $a(z)$ satisfy (2) this is true for $d \geq 3$. In the case when the transition intensities $a(z)$ satisfy (3) this is true for $d = 1$ and $\alpha \in (0, 1)$, or for $d \geq 2$ and $\alpha \in (0, 2)$.

Another possible case is when $G_\lambda(0, 0) \rightarrow \infty$ as $\lambda \rightarrow 0$. For positive matrices, the leading eigenvalue always exceeds any diagonal element of the matrix. Therefore, in this case $\gamma_0(\lambda) \geq \Gamma_{11}(\lambda) = G_\lambda(0, 0)$, from which $\gamma_0(\lambda) \rightarrow \infty$ as $\lambda \rightarrow 0$. Then $\beta_c = 0$. If the transition intensities $a(z)$ satisfy (2) this is true for $d = 1$ or $d = 2$. If the transition intensities $a(z)$ satisfy (3) this is true for $d = 1$ and $\alpha \in [1, 2)$. By Lemma 5 here again $\beta_{c1} \geq \frac{1}{\gamma^*} > 0$. This situation is illustrated in Fig. 1(b).

By Lemma 1 the eigenvalues λ_i of the operator \mathcal{H}_β are solutions of the equations (5), see Fig. 1. However, since by Lemma 3 every function $\gamma_i(\lambda)$ is strictly decreasing when $\lambda \geq 0$, then the total number of the solutions $\lambda_i(\beta)$ of these equations does not exceed N .

From inequalities (21) it follows that if the operator \mathcal{H}_β has positive eigenvalues for some fixed β , then the maximal of them is $\lambda_0 = \lambda_0(\beta)$ which is a solution of the equation

$$\gamma_0(\lambda_0) = \frac{1}{\beta}, \quad (22)$$

is simple and strictly exceeds others. The minimum value of the solution λ_0 of equation (22) is $\lambda_0 = 0$. The corresponding value β_c of the parameter β is critical. Since the function $\gamma_0(\lambda)$ is strictly decreasing then the eigenvalue $\lambda_0 = \lambda_0(\beta)$ increases with the increase of parameter β . The theorem is proved. \square

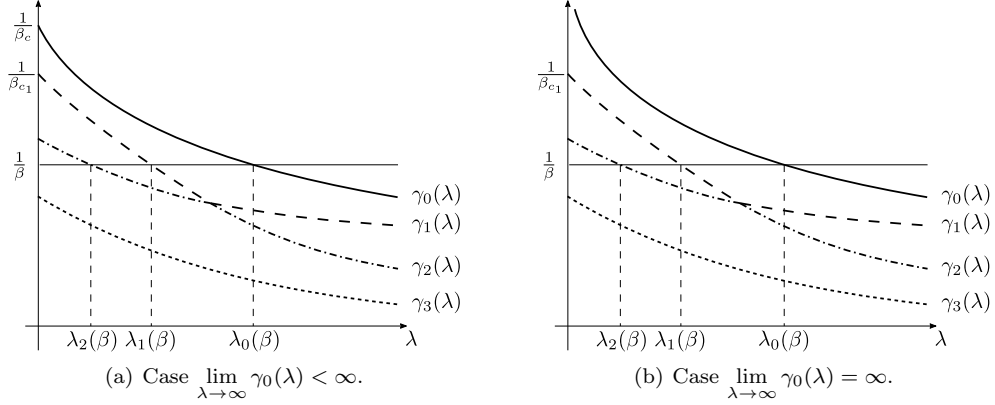


Figure 1: Plots of the functions $\gamma_i(\lambda)$.

5 Multiple Eigenvalues: an Example

By Theorem 2 the principal eigenvalue $\lambda_0(\beta)$ of the operator \mathcal{H}_β is always simple. In this section we demonstrate that some other eigenvalues $\lambda_1(\beta), \dots, \lambda_{N-1}(\beta)$ of this operator may actually coincide (i.e. their multiplicity may be greater than one) and this situation is possible even in the case of arbitrary finite number of sources (of equal intensity). As is shown in the following Example 1 this situation may occur if there is a certain ‘symmetry’ of the spatial configuration of the sources x_1, x_2, \dots, x_N .

In Example 1 we assume that the function of transition probabilities is symmetric in the following sense: its values do not change at any permutation of arguments. In particular, a function of a vector variable z is symmetric if its values are the same at any permutation of coordinates of vector z .

Let us present a statement to be used further for ‘integrable’ models, where equations for estimating λ_i from Theorem 2 can be found explicitly.

Lemma 6. *If the function of transition probabilities $a(z)$ is symmetric in the following sense: its values do not change at any permutation of arguments, then the function $G_\lambda(z)$ is also symmetric in the same sense.*

Proof. Let us recall that Green’s function $G_\lambda(z)$ can be represented [Yarovaya, 2007] in the form (8), where $\phi(\theta)$ is the Fourier transform of the function of transition probabilities $a(z)$ and is defined by $\phi(\theta) = \sum_{z \in \mathbb{Z}^d} a(z) e^{i(\theta, z)}$, $\theta \in [-\pi, \pi]^d$. To prove the lemma it suffices to demonstrate that $G_\lambda(z) = G_\lambda(\mathbf{R}z)$ for every permutation matrix \mathbf{R} (i.e. a matrix whose rows and columns have the only one nonzero element, which is equal to one). So,

$$\begin{aligned} \phi(\mathbf{R}\theta) &= \sum_{z \in \mathbb{Z}^d} a(z) e^{i(\mathbf{R}\theta, z)} = \sum_{z \in \mathbb{Z}^d} a(z) e^{i(\theta, \mathbf{R}^* z)} \\ &= \sum_{z' \in \mathbb{Z}^d} a((\mathbf{R}^*)^{-1} z') e^{i(\theta, z')} = \sum_{z' \in \mathbb{Z}^d} a(z') e^{i(\theta, z')} = \phi(\theta), \end{aligned}$$

where the equality $a((\mathbf{R}^*)^{-1} z') = a(z')$ holds for all $z' \in \mathbb{Z}^d$ since the function $a(z)$ is symmetric. Hence the function $\phi(\theta)$ is also symmetric. Further,

$$\begin{aligned} G_\lambda(\mathbf{R}z) &= \frac{1}{(2\pi)^d} \int_{[-\pi, \pi]^d} \frac{e^{i(\theta, \mathbf{R}z)}}{\lambda - \phi(\theta)} d\theta = \frac{1}{(2\pi)^d} \int_{[-\pi, \pi]^d} \frac{e^{i(\mathbf{R}^* \theta, z)}}{\lambda - \phi(\theta)} d\theta \\ &= \frac{1}{(2\pi)^d} \int_{[-\pi, \pi]^d} \frac{e^{i(\theta, z)}}{\lambda - \phi((\mathbf{R}^*)^{-1} \theta)} d\theta = G_\lambda(z) \end{aligned}$$

for every permutation matrix \mathbf{R} , where the equality $\phi(\theta) = \phi((\mathbf{R}^*)^{-1} \theta)$ again holds for all $\theta \in [-\pi, \pi]^d$ since the function $\phi(\theta)$ is symmetric. Thus, the function $G_\lambda(z)$ is also symmetric. \square

Now, we are ready to present an example.

Example 1. Let the function of transition probabilities $a(z)$ be symmetric in the following sense: its values do not change at any permutation of arguments, and let x_1, \dots, x_N , where $N \geq 2$, be the vertices of a regular simplex (i.e. of a simplex for which the lengths of the edges are equal). For example, we can take

$$x_1 = \{1, 0, \dots, 0\}, \quad x_2 = \{0, 1, \dots, 0\}, \quad \dots, \quad x_N = \{0, 0, \dots, 1\}.$$

By Lemma 1 the existence of a non-trivial solution λ of the equation (5) for some β is equivalent to the resolvability of the equation

$$\det \left(\Gamma(\lambda) - \frac{1}{\beta} I \right) = 0, \quad (23)$$

where the matrix $\Gamma(\lambda)$ with the elements $\Gamma_{ij}(\lambda) = G_\lambda(x_i, x_j)$ is defined by equation (6).

Since the random walk by assumption is symmetric and homogeneous then

$$G_\lambda(x_i, x_j) = G_\lambda(0, x_i - x_j) = G_\lambda(0, x_j - x_i) = G_\lambda(x_j - x_i).$$

From the definition of the function $G_\lambda(u, v) \equiv G_\lambda(u - v)$ by Lemma 6 it follows that all the values $G_\lambda(x_j - x_i)$ coincide with each other when $i \neq j$, and hence they coincide with $G_\lambda(x_1 - x_2) = G_\lambda(z_*)$ (for simplicity we denote $z_* = x_1 - x_2$). So,

$$G_\lambda(x_i, x_j) = G_\lambda(x_j - x_i) \equiv G_\lambda(x_1 - x_2) = G_\lambda(z_*) \quad \text{for all } i \neq j, \quad (24)$$

while for $i = j$ we have $G_\lambda(x_i, x_i) \equiv G_\lambda(x_i - x_i) = G_\lambda(0) = G_\lambda$. Thus, we can represent equation (23) as

$$\det \begin{bmatrix} G_\lambda - \frac{1}{\beta} & \cdots & G_\lambda(z_*) \\ G_\lambda(z_*) & \cdots & G_\lambda(z_*) \\ \cdots & \cdots & \cdots \\ G_\lambda(z_*) & \cdots & G_\lambda - \frac{1}{\beta} \end{bmatrix} = 0.$$

Using a bit cumbersome but standard linear transforms we rewrite the last determinantal equation as

$$\left(G_\lambda - G_\lambda(z_*) - \frac{1}{\beta} \right)^{N-1} \det \begin{bmatrix} G_\lambda - \frac{1}{\beta} + (N-1)G_\lambda(z_*) & \cdots & G_\lambda(z_*) \\ 0 & \cdots & 0 \\ \cdots & \cdots & \cdots \\ 0 & \cdots & -1 \end{bmatrix} = 0,$$

which is equivalent to

$$\left(G_\lambda + (N-1)G_\lambda(z_*) - \frac{1}{\beta} \right) \left(G_\lambda - G_\lambda(z_*) - \frac{1}{\beta} \right)^{N-1} = 0.$$

From this last equation, it is seen that the operator \mathcal{H}_β has a simple leading eigenvalue $\lambda = \lambda_0(\beta)$ satisfying the equation

$$G_\lambda + (N-1)G_\lambda(z_*) - \frac{1}{\beta} = 0,$$

and an eigenvalue $\lambda = \lambda_1(\beta) = \dots = \lambda_{N-1}(\beta)$ of multiplicity $N-1$ satisfying the equation

$$G_\lambda - G_\lambda(z_*) - \frac{1}{\beta} = 0.$$

Hence in this case the quantities β_c and β_{c_1} can be calculated explicitly:

$$\beta_c = \frac{1}{G_0 + (N-1)G_0(z_*)}, \quad \beta_{c_1} = \frac{1}{G_0 - G_0(z_*)}. \quad (25)$$

Remark 3. Under the conditions of Example 1 according to (24) and (25) the quantity β_{c_1} depends on the norm $|z_*|$ of the vector z_* (i.e. on the distance between the sources) and does not depend on the number N of the sources, that is $\beta_{c_1} = \beta_c(|z_*|) > 0$. At the same time the quantity β_c depends not only on the distance between the sources but also on the number N of the sources, that is $\beta_c = \beta_c(|z_*|, N)$, and in such a way that $\beta_c(|z_*|, N) \rightarrow 0$ as $N \rightarrow \infty$ when z_* is fixed. Moreover, $\beta_c(|z_*|, N) \equiv 0$ when $G_0 = \infty$.

Remark 4. For the case when $\mathcal{A} = \kappa\Delta$, $\kappa > 0$, is the lattice Laplacian, Example 1 was presented in [Yarovaya, 2015], and under assumption of finite variance of jumps it was given in [Antonenko and Yarovaya, 2015].

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